Paradox at Every Scale

Yablo's sequence sieved, quotiented, and made self-similar

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Abstract

Which patterns of reference without self-reference are paradoxical? Yablo's sequence shows there are some; characterising them has stayed open because the known tools say only what a paradox must contain, not what makes it one. This paper introduces *sieves*—self-similar patterns in which each sentence denies the sentences at a fixed set of distances—and shows that on them the question can be answered outright. A periodic sieve is paradoxical exactly when a finite quotient of its reference digraph has no kernel, so the circularity the non-circular paradox was accused of hiding becomes a literal odd cycle; for every sieve that can be written down, paradoxicality is decidable, though for digraphs in general it is not; and the size of the sieve is irrelevant, only its structure counts. A sequence left open in 2002 is settled, the ω-inconsistency of the Yablo biconditionals is explained as the absence of a kernel, and Yablo's sequence is shown to be fixed-point circular like a fractal, not reference circular.

1 The sequence and the question

Consider an infinite sequence of sentences $S_1, S_2, S_3, \ldots$, each to the effect that every subsequent sentence is untrue:

$$\begin{aligned} (S_1) &\quad \text{for all } k > 1,\ S_k \text{ is untrue,} \\ (S_2) &\quad \text{for all } k > 2,\ S_k \text{ is untrue,} \\ (S_3) &\quad \text{for all } k > 3,\ S_k \text{ is untrue,} \\ &\quad \vdots \end{aligned}$$

If some S_n is true, then every later sentence is untrue, and in particular S_{n+1} is; but everything after S_{n+1} is untrue as well, which is exactly what S_{n+1} says, so S_{n+1} is true after all. Hence no S_n is true. But then everything after S_1 is untrue, which is what S_1 says, so S_1 is true. This is Yablo's paradox (1985; 1993). No sentence in it refers to itself, nor to any sentence that refers back to it; reference runs forward only, and yet there is no consistent way to assign truth-values.

The sequence raised a question that thirty years of discussion have not settled: what, if not circularity, is the structural feature of a pattern of reference that makes it paradoxical? The discussion has run in three channels. Philosophers have asked whether the sequence is circular after all, since it can only be given all at once by a predicate that is a fixed point of an operator built from that predicate. Logicians have shown that in arithmetic with a truth predicate the sequence is consistent but ω-inconsistent, and have mapped the principles responsible. Graph theorists have asked which digraphs are *dangerous*, in the sense that some assignment of sentences to their vertices is paradoxical, have proved that a dangerous acyclic digraph must have infinitely many vertices of infinite out-degree, and have conjectured that every such digraph contains Yablo's as a minor. What has been lacking is a family of reference patterns, large enough to contain Yablo's sequence and its natural relatives and structured enough that the question can be *computed* on it.

This paper supplies such a family. A *sieve* is a set R of distances, or more generally of words over a finite alphabet, and the *sieve system* lets each sentence say that the sentences at R-distance from it are untrue. Yablo's sequence is the sieve of all positive distances. Every sieve system is *self-similar*: the sentences below any given sentence form a copy of the whole system. The main results are that paradoxicality of a sieve system is governed by a finite quotient of its reference digraph when the sieve is periodic, by an algebraic absorption property of the sieve in general, and that it is decidable whenever the sieve can be given to an algorithm. The circularity that the non-circular paradox was accused of hiding turns out, for periodic sieves, to be an odd cycle in that finite quotient. The results are then applied to the three channels: to the arithmetic of truth, where they generalise Ketland's ω-inconsistency theorem; to self-similar fractals, where they make exact the sense in which Yablo's sequence is a fixed point; and to the circularity debate.

Literature

The circularity debate began when Priest (1997) observed that all the sentences can be given at once only through a predicate that the diagonal lemma supplies as a fixed point of an operator built from that very predicate. It ran through Sorensen (1998), Beall (2001), Bueno and Colyvan (2003a,b), Ketland (2004, 2005), Leitgeb (2002), Cook (2006), Schlenker (2007), and the symposium in Barrio (2012), Picollo (2012) and Ojea (2012); it was given a book by Cook (2014); it continues in Butler (2018), Halbach and Zhang (2017), Çevik (2020) and Hsiung (2020). Yablo's own later view is in Yablo (2006).

The formal strand begins with the observation that in arithmetic with a truth predicate the sequence is consistent but ω-inconsistent. Ketland (2005), Leitgeb (2001), Barrio (2010), Picollo (2013), Barrio and Picollo (2013) and Godziszewski and Urbaniak (2026) determine which principles give consistency, which give ω-inconsistency, and which, in second-order arithmetic, give unsatisfiability; Cieśliński and Urbaniak (2013), Leach-Krouse (2014), Kurahashi (2014) and Cheng (2020) replace truth by provability and obtain incompleteness theorems; Halbach and Zhang (2017), Karimi (2019a) and Salehi (2020) strip the construction of arithmetic altogether and find a statement about an arbitrary binary relation, which the last of these writes outright as "the relation has no kernel".

The graph-theoretic strand, to which this paper belongs, begins with the pointer networks of Gaifman (1992). Cook (2004) put the Liar and Yablo's sequence into one infinitary language, observed that the acceptable valuations of a system of mutual denials are the kernels of its reference digraph, and showed how to *unwind* any finite system into an infinite non-circular one. Bezem et al. (2012), Dyrkolbotn and Walicki (2014), Walicki (2017, 2019, 2024) and Bodanza (2023) made the kernel reading general and connected it to the combinatorics of infinite digraphs. Rabern et al. (2013), Rabern and Rabern (2021), Beringer and Schindler (2016, 2017), Beringer (2025) and Hsiung (2021, 2025) studied dangerous digraphs, proved that a dangerous acyclic digraph has infinitely many vertices of infinite out-degree, and conjectured that it must contain Yablo's digraph as a finitary minor. Bolander (2024) and Cook (n.d.) survey the field.

The pattern has many variants: with Curry's conditional in place of negation (Beall, 1999; Cook, 2009), with "some" in place of "all" (Sorensen, 1998), with "infinitely many" and "all but finitely many" (Schlenker, 2007; Cook, 2017), with the two quantifiers mixed in one sequence (Cook, 2015); in set theory (Shen, 1953; Goldstein, 1994; Garti, 2021), in the theory of denotation (Uzquiano, 2004), for Grelling's and Zwicker's paradoxes (Martínez Fernández and Valor Abad, 2014), for belief (Başkent, 2018), for knowledge and the preface (Sorensen, 1998), for provability (Cieśliński and Urbaniak, 2013; Leach-Krouse, 2014; Kurahashi, 2014), and as theorems of temporal logic (Karimi and Salehi, 2014; Karimi, 2019b). Jongeling et al. (2002), early and little cited, put the sequence on $\omega + 1$ and on a tree of decimal numerals; both experiments return below.

Results

§2 collects what is needed about kernels. A classical valuation of any system in which each sentence says that all its referents are untrue is a kernel of the reference digraph (Lemma 2.1); the restriction to such *uniform* systems is shown to be substantive, since with mixed signs the reference digraph no longer determines paradoxicality (Proposition 2.2); Yablo's argument becomes a statement about partial orders (Proposition 2.3); and Richardson's theorems show that without an odd cycle of reference, and with either finitely many referents per sentence or a bipartite reference graph, there is no paradox (Theorem 2.4).

§3 defines sieves and proves the general results about them. An absorbing word in the sieve forces paradox (Theorem 3.2), and this condition is the free-monoid form of a first-order condition of Karimi and Salehi; on $\mathbb{N}$ a ray in the sieve forces paradox (Lemma 3.4); a sieve whose members share a 2-adic valuation never does (Proposition 3.5). The examples show that size is irrelevant: the primes and the powers of two are harmless, the even numbers are paradoxical. A tree-indexed sequence from 2002 whose authors gave no verdict is settled, and a tree sieve is exhibited that contains a copy of Yablo's sequence below every vertex and yet has a valuation (Proposition 3.11).

§4 proves the central theorem: the valuations of a periodic sieve on $\mathbb{N}$ are exactly the lifts of the kernels of a finite circulant digraph (Theorem 4.1). Every paradoxical periodic sieve therefore has an odd cycle in its quotient, the sieve of distances $\equiv 1 \pmod 3$ is a directed triangle in disguise, and the periodic sieves are completely classified. The lifting half of the theorem survives on trees (Proposition 4.7).

§5 proves that paradoxicality is decidable for eventually periodic sieves on $\mathbb{N}$ and for regular sieves on trees, with periodic and regular valuations as witnesses (Theorem 5.1), against a background in which kernel existence for recursive digraphs is Σ^1_1-complete; for finite sieves every valuation is purely periodic (Corollary 5.2).

§6 places the sieves in arithmetic with a truth predicate. Ketland's theorem generalises: the local theory of any sieve is consistent, it has a standard model exactly when the sieve has a kernel, and when the paradox is provable the theory is ω-inconsistent and its uniform version inconsistent (Theorem 6.1).

§7 realises the sieve systems on the cells of a self-similar fractal and compares the fixed point supplied by the diagonal lemma with the one supplied by Hutchinson's theorem. §8 draws the consequences for the circularity debate, and §9 lists open problems.

Conventions

"Untrue" means "not true"; classically this is "false", and the two words are used interchangeably. A *classical valuation* of a system of sentences is an assignment of truth-values that makes each sentence true exactly when what it says is the case; a system is *paradoxical* if it has no classical valuation. $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Yablo's sequence is indexed from 1 when it is quoted and from 0 when it is computed with. "S_n refers to S_k", written $n \to k$, means that S_n speaks about S_k.

2 Kernels

A *reference digraph* is a directed graph $\mathscr{D} = (V, E)$, finite or infinite, with loops allowed. Each vertex x carries a sentence S_x saying that everything x points to is untrue:

$$(S_x) \quad \text{for every } y \text{ with } x \to y,\ S_y \text{ is untrue.} \tag{1}$$

The system of these sentences is the *uniform system* $\mathscr{U}(\mathscr{D})$ on $\mathscr{D}$. A vertex with no out-neighbour carries a vacuously true sentence, and a vertex with a loop carries a Liar. Yablo's sequence is $\mathscr{U}(\mathscr{D})$

for $\mathscr{D} = (\omega, <)$. A *classical valuation* of $\mathscr{Y}(\mathscr{D})$ assigns truth-values so that S_x is true iff every S_y with $x \to y$ is false, and the system is *paradoxical* if it has no such valuation.[1]

A set $K \subseteq V$ is a *kernel* of $\mathscr{D}$ if it is *independent*, meaning that no edge joins two members of K and no member carries a loop, and *absorbing*, meaning that every vertex outside K has an edge into K. Kernels were introduced by von Neumann and Morgenstern (1944) as the solutions of games, and they have a literature of their own (Berge, 1985; Boros and Gurvich, 2006; Goldfeder and Tecpa-Galván, 2025).

Lemma 2.1 (Kernel lemma)**.** *The classical valuations of $\mathscr{Y}(\mathscr{D})$ correspond exactly to the kernels of $\mathscr{D}$, via $K = \{x : S_x \text{ is true}\}$. In particular, $\mathscr{Y}(\mathscr{D})$ is paradoxical iff $\mathscr{D}$ has no kernel.*

Proof. Read (1) as a condition on K: $x \in K$ iff no out-neighbour of x lies in K. The forward direction of this biconditional is independence and the backward direction is absorption. □

The lemma is due to Cook (2004), who proved it in an appendix for digraphs without sinks. The general form is in Bezem et al. (2012) and Dyrkolbotn and Walicki (2014), who read any infinitary propositional theory as a digraph in "graph normal form" and a paradox as a digraph without a kernel. Walicki (2017, 2024) build proof systems on that reading, Beringer (2025) removes Cook's restriction to sink-free digraphs, Bodanza (2023) relaxes the treatment of sinks, and Salehi (2020) writes Yablo's paradox as the second-order sentence "R has no kernel" and proves that this sentence is not equivalent to any first-order one.[2]

Why uniform systems

The restriction to systems in which every reference is a denial is substantive: once truth and falsity are both ascribed, the reference digraph no longer determines whether a system is paradoxical. Interleave two sequences $A_1, B_1, A_2, B_2, \ldots$ and let the A's say that all later B's have truth-value ε_A, the B's that all later A's have truth-value ε_B:

$$(A_n) \quad \text{for all } k \geq n,\ B_k \text{ has value } \varepsilon_A,$$
$$(B_n) \quad \text{for all } k > n,\ A_k \text{ has value } \varepsilon_B.$$

Proposition 2.2. *The interleaved system is paradoxical iff $\varepsilon_A \neq \varepsilon_B$.*

Proof. If both values are "untrue", making every A_n true and every B_n false is a valuation: each A_n correctly reports that all B's are untrue, and each B_n is untrue because some later A_k is true. If both values are "true", making every sentence true is a valuation. Suppose ε_A is "untrue" and ε_B is "true". If some B_n were true, then A_{n+1} would be true, so every B_k with $k \geq n+1$ would be untrue, in particular B_{n+1}, so some A_m with $m > n+1$ would be untrue, contradicting B_n. So every B_n is untrue; but then every A_n is true, whence every B_n is true. The remaining case is symmetric: if some A_n is true then B_n is true, so A_{n+1} is untrue, so some B_k with $k \geq n+1$ is untrue, contradicting A_n; so every A_n is untrue, every B_n is true, and every A_n is true. □

All four systems have the same reference digraph. In the terms of Cook (2004), they are the *unwindings*, in the interleaved order, of the four two-sentence systems with one reference in each direction: the no-no pair {"S_2 is untrue", "S_1 is untrue"} unwinds to a harmless system, Jourdain's

[1] This is the notion of paradoxicality used throughout the graph-theoretic literature (Cook, 2004; Rabern et al., 2013; Beringer and Schindler, 2017). It does not prejudge what one should do about a paradox, whether to go paraconsistent with Priest (1997), paracomplete with Field (2008), or three-valued as Gaifman (1992), Dyrkolbotn and Walicki (2014) and Walicki (2024) do; it records only that classical truth-values cannot be found.

[2] Conventions differ between the two literatures. In game theory the edges often point *toward* the dominating vertex, and Richardson's "solutions" are kernels of the reversed digraph. Throughout this paper $x \to y$ means that S_x refers to S_y, so that Yablo's digraph is the transitive tournament on ω.

card {"S_2 is untrue", "S_1 is true"} to a paradoxical one. The proposition is thus a corollary of Cook's theorem that unwinding preserves paradoxicality, which in kernel language is the theorem of Bezem et al. (2012, Theorem 3.9) that a digraph has a kernel iff its layered lifting does, and in its most general form is the unwinding of a Boolean net along a definable digraph of Hsiung (2021).[3] For uniform systems the digraph is the whole story, and the rest of the paper is about uniform systems.

Orders

The first instance of the kernel method is Yablo's own argument, which works on any partial order.

Proposition 2.3. *Let $(P,<)$ be a non-empty strict partial order and $\mathscr{D}_P = (P,<)$, so that S_x says that every sentence above x is untrue. Then $\mathscr{Y}(\mathscr{D}_P)$ has a classical valuation iff every element of P is maximal or lies below a maximal element. When a valuation exists it is unique: the maximal sentences are true and all others are false.*

Proof. Let M be the set of maximal elements. Every kernel K contains M, because a maximal element has no out-neighbour and absorption forces it into K. Every kernel is contained in M: if $k \in K$ and $k < y$, then $y \notin K$ by independence, so by absorption some $k' \in K$ has $y < k'$, and then $k < k'$ is an edge inside K. The set M is independent, and it is absorbing exactly when every non-maximal element lies below a maximal one. □

For a linear order the condition is that a greatest element exists. For an ordinal α it is that α is not a non-zero limit: the sequence of order type $\omega+1$ is not paradoxical, as Jongeling et al. (2002, Example 4) observed, while the sequences of order types ω, $\omega+\omega$, ω_1, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are. The paradoxical half of the proposition is known in several forms. Ketland (2005) observed that the scheme $\forall x\,[\varphi(x) \leftrightarrow \forall y\,(\psi(x,y) \to \neg\varphi(y))]$ is refutable in first-order logic from the seriality and transitivity of ψ alone, and the elements above an element with no maximal element above it form a serial transitive sub-order closed under reference; Cook (2014, Theorem 1.3.1) proves the result for "Yabloesque chains" on ω, and Karimi and Salehi (2014) for the temporal operator "always"; Bezem et al. (2012, Prop. 5.9) and Dyrkolbotn and Walicki (2014, §4.2.4) note that a kernel of a weakly complete digraph, one in which of any two vertices one points to the other, must be a single vertex that everything else points to; and Schlechta (2018, Fact 4.3) gives the criterion for transitive digraphs with cycles allowed, namely that the uniform system is paradoxical iff some vertex has successors all of which have successors. Proposition 2.3 is the irreflexive case of Schlechta's criterion with the uniqueness of the valuation added.

Richardson's theorems

The transitive tournament on ω is the standard example showing that the theorem of Richardson (1953), that a finite digraph with no odd directed cycle has a kernel, does not extend to infinite digraphs without further hypotheses; Walicki (2019) calls it the fundamental unsafe dag.[4] Two extensions do hold.

Theorem 2.4 (Richardson)**.** *(a) If $\mathscr{D}$ has no odd directed cycle and every vertex has finitely many out-neighbours, then $\mathscr{D}$ has a kernel.*

(b) If every undirected cycle of $\mathscr{D}$ has even length, that is, if the underlying undirected graph of $\mathscr{D}$ is bipartite, then $\mathscr{D}$ has a kernel.

[3] Cook (2015) mixes quantifiers rather than signs, letting each sentence say either that all later sentences are untrue or that some later sentence is; paradox occurs exactly when one of the two kinds occurs only finitely often. The no-no pair has two classical valuations, and Sorensen (2001) and Cook (2011) argue that it is paradoxical nonetheless, because each valuation breaks a symmetry that the sentences do not; the interleaved system inherits this feature.

[4] Richardson (1953) precedes Yablo (1985) by three decades, and in the same year Shen (1953) published the paradox of the class of all grounded classes, whose engine is likewise an infinite descending chain rather than a circle.

Part (a) is stated in this form, and credited to Richardson (1953), by Walicki (2019, Theorem 1.1). It follows from a compactness principle that Galeana-Sánchez and Hernández-Cruz (2014) attribute to Duchet and Meyniel (1993): a digraph of finite out-degree has a kernel, and indeed every induced subdigraph has one, iff every *finite* induced subdigraph has one; and a finite induced subdigraph of a digraph without odd cycles has no odd cycles. For acyclic digraphs the compactness argument is given in Bezem et al. (2012, §4.1) and Dyrkolbotn and Walicki (2014, §4.1.1). Part (b) is Richardson's older theorem (Richardson, 1946), reproved without his antisymmetry hypothesis by Isbell (1957); it needs no finiteness at all.

Corollary 2.5. *If every sentence of a uniform system refers to finitely many sentences and the reference relation has no odd cycle, then the system has a classical valuation.*

The corollary is known in stronger forms. Rabern et al. (2013, pp. 751, 754) show by compactness that an acyclic digraph of finite out-degree is not dangerous, so that no assignment of sentences to its vertices, uniform or not, is paradoxical, and that a dangerous acyclic digraph has infinitely many vertices of infinite out-degree; Rabern and Rabern (2021) give a topological proof; Hsiung (2020) proves the first-order version by König's lemma, Hsiung (2021) the version for unwindings, and Hsiung (2025) shows that the vertices of infinite out-degree can be found infinitely often along a single ray. Over RCA_0 the statement that every finitely branching dag has a kernel is equivalent to weak König's lemma (Bezem et al., 2012), and for digraphs with no infinite walk the existence and the uniqueness of the kernel are each equivalent to ATR_0 (Friedman, 2010). The earliest forms of the observation are the remark of Hardy (1995) that no finite set of Yablo's sentences is contradictory, developed by Forster (2004) and Ketland (2004) and made into the slogan "circular in the limit" by Çevik (2020), and the proof by Jongeling et al. (2002) that an infinite sequence of *finite* propositions without self-reference cannot be paradoxical.

The hypothesis of Theorem 2.4(a) fails for Yablo's digraph in one way only: every vertex has infinitely many out-neighbours. The question the rest of the paper answers, for self-similar systems, is what those infinitely many referents must look like for the paradox to go through.[5]

3 Sieves

Let Σ be a finite alphabet and let Σ^* be the free monoid of finite words over Σ; for a one-letter alphabet $\Sigma^* \cong \mathbb{N}$ and concatenation is addition. A *sieve* is a non-empty set $R \subseteq \Sigma^+$ of non-empty words. The reference digraph $\mathscr{D}(\Sigma, R)$ has vertex set Σ^* and an edge $w \to wg$ for each $g \in R$, and the *sieve system* $\mathscr{Y}(\Sigma, R)$ is the uniform system on it:

$$(S_w) \quad \text{for every } g \in R,\ S_{wg} \text{ is untrue.} \tag{2}$$

On $\mathbb{N}$, with $R \subseteq \mathbb{N}^+$, the sentences read

$$(S_n) \quad \text{for every } r \in R,\ S_{n+r} \text{ is untrue,} \tag{3}$$

and $\mathscr{D}(\mathbb{N}, R)$ has an edge $n \to n+r$ for each $r \in R$. Yablo's sequence is the sieve $\mathbb{N}^+$. Every sieve digraph is acyclic, since an edge increases length, and every sieve system is *self-similar*: for any word u the map $w \mapsto uw$ embeds $\mathscr{D}(\Sigma, R)$ isomorphically onto the subgraph of vertices below u, and if K is a kernel then so is $u^{-1}K = \{w : uw \in K\}$.

Two sieves on the binary tree $\{0,1\}^*$ fix ideas. Writing $\tau \sqsupset \sigma$ when σ is a proper prefix of τ, the sieve $\{0,1\}^+$ gives the system in which every sentence says that everything below it is untrue,

$$(S_\sigma) \quad \text{for all } \tau \sqsupset \sigma,\ S_\tau \text{ is untrue,} \tag{4}$$

[5] For arbitrary digraphs the question is open. Walicki (2019) conjectures that odd cycles and rays dominated by infinitely many of their own vertices are the only obstructions to a kernel, and proves this for digraphs with finitely many ends; a claimed proof of the general statement (Walicki, 2023b) was withdrawn. The tree sieves of §3 have uncountably many ends and so lie outside the proved case.

and the sieve $\{0,1\}$ gives the system in which every sentence speaks only of its two children,

$$(S_\sigma) \quad S_{\sigma 0} \text{ is untrue and } S_{\sigma 1} \text{ is untrue.} \tag{5}$$

The second is not paradoxical: making the sentences at even depth true and those at odd depth untrue is a valuation, and by Theorem 2.4 a valuation had to exist, both because every sentence refers to two others and because the underlying graph is a tree. By Rabern et al. (2013, Theorem 24), no assignment of sentences whatever to the vertices of a forest is paradoxical. The first is paradoxical, as the next theorem shows.

Absorption

Write $N^+(x)$ for the set of out-neighbours of x.

Theorem 3.1. *Suppose $\mathscr{D}$ is non-empty and every vertex x has an out-neighbour y with $N^+(y) \subseteq N^+(x)$. Then $\mathscr{D}$ has no kernel.*

Proof. Let K be a kernel, let $x \in K$, and take y as in the hypothesis. Then $y \notin K$, so some $z \in N^+(y)$ lies in K; but $z \in N^+(x)$, so $x \to z$ is an edge inside K. Hence K is empty. An empty set absorbs nothing, and every vertex has an out-neighbour, so K is not a kernel. □

The argument is the first half of Yablo (1993): S_n refers to S_{n+1}, and everything that S_{n+1} refers to, S_n refers to as well. Ketland (2005) isolated the transitive case, since a serial transitive relation satisfies the hypothesis, and Halbach and Zhang (2017) derive the paradox from seriality and transitivity without the diagonal lemma. Karimi (2019a) weakens their order axioms to exactly the hypothesis of Theorem 3.1, under the name W-TRANS, and Salehi (2020, Theorem 1) shows that this hypothesis is the first of a strictly decreasing chain of first-order conditions $\forall x\, \theta_n(x)$, each sufficient for the absence of a kernel and none necessary. On a sieve the hypothesis becomes a property of words. Call $g \in R$ *absorbing* for R if $gR \subseteq R$.

Theorem 3.2 (Absorption)**.** *If some $g \in R$ is absorbing for R, then $\mathscr{Y}(\Sigma, R)$ is paradoxical.*

Proof. The out-neighbourhood of w is wR, and $N^+(wg) = wgR \subseteq wR$ exactly when $gR \subseteq R$. Apply Theorem 3.1 with $y = wg$. □

Corollary 3.3. *If R is a subsemigroup of Σ^+, or equivalently if the reference relation of $\mathscr{Y}(\Sigma, R)$ is transitive, then $\mathscr{Y}(\Sigma, R)$ is paradoxical. In particular the system (4) on the full m-ary tree is paradoxical for every $m \geq 1$, and so is its restriction to any subtree in which every vertex has a descendant.*

Proof. Every element of a subsemigroup is absorbing. For a subtree in which every vertex has a descendant, apply Theorem 3.1 directly: a vertex σ and any child σa of it in the subtree satisfy $N^+(\sigma a) \subseteq N^+(\sigma)$. □

The corollary covers $\mathbb{N}^+$, the words of even length, the words of length at least two, the words beginning with a fixed letter, and sieves with no arithmetical description, such as the binary words that begin and end with 0 and never contain 11, whose number of length n grows only like φ^n for φ the golden ratio. Absorption is weaker than transitivity, which would require $RR \subseteq R$; one absorbing word suffices. On a sieve digraph, moreover, Salehi's chain of conditions collapses to its first term, because each θ_n quantifies only forward along R, all forward cones are isomorphic, so each θ_n has the same truth-value at every vertex, and since every vertex has a successor the condition θ_{n+1} reduces to θ_n. Sharper criteria for sieves therefore have to use something other than Salehi's hierarchy; those of §§4 and 5 use periodicity and monadic second-order logic.

Rays

Lemma 3.4 (Rays)**.** *If $R \subseteq \mathbb{N}^+$ contains all positive multiples of some $g \geq 1$, then $\mathscr{Y}(\mathbb{N},R)$ is paradoxical.*

Proof. Let K be a kernel and $n \in K$. Then $n+g, n+2g, \ldots \notin K$, so no two members of K lie in the same residue class modulo g, and K has at most g elements. Any m larger than all of them is outside K, and its referents are larger still, so m is not absorbed. □

On $\mathbb{N}$ absorption implies the hypothesis of the lemma, since $g+R \subseteq R$ gives $g, 2g, 3g, \cdots \in R$, but not conversely: the sieve $\{1\} \cup \{2,4,6,\ldots\}$ contains the ray $2\mathbb{N}^+$, and no element of it is absorbing.[6] Nor is the lemma a characterisation, as Example 4.3 below shows.

Bipartite sieves

Proposition 3.5. *The underlying undirected graph of $\mathscr{D}(\mathbb{N},R)$ is bipartite iff all members of R have the same 2-adic valuation, that is, iff $R \subseteq 2^k \cdot \{1,3,5,\ldots\}$ for some k. Every such sieve has a classical valuation.*

Proof. If every $r \in R$ is 2^k times an odd number, colour n by the parity of $\lfloor n/2^k \rfloor$; every edge $n \to n+r$ changes the colour. If $r,s \in R$ have different valuations, let $d = \gcd(r,s)$; then s/d steps of size r upward followed by r/d steps of size s downward form a closed walk of length $(r+s)/d$, which is odd, and a closed walk of odd length contains an odd cycle. The second sentence is Theorem 2.4(b). □

The odd sieve $\{1,3,5,\ldots\}$ is the simplest case, with the even numbers as the valuation; the sieves of odd primes, of odd squares, and of numbers congruent to 2 modulo 4 are harmless for the same reason. On a tree the same colouring, by the parity of the length of a word, gives a valuation whenever every word of R has odd length.

Examples

Each claim below is established by exhibiting a kernel or by one of the results above, and each has been checked mechanically.[7]

Example 3.6 (The even sieve)**.** $R = \{2,4,6,\ldots\}$ is paradoxical, by Lemma 3.4 or by Corollary 3.3; it consists of two interleaved copies of Yablo's sequence that never speak to each other.

Example 3.7 (All but one)**.** $R = \mathbb{N}^+ \setminus \{2\}$ is paradoxical, since $3+R \subseteq R$.

Example 3.8 (Powers of two)**.** $R = \{1,2,4,8,\ldots\}$ is not paradoxical: make the multiples of 3 true. Since $2^k \equiv 1$ or $2 \pmod 3$, a multiple of 3 never sees another; and if $n \equiv 1$ then $n+2$, while if $n \equiv 2$ then $n+1$, is a multiple of 3.

Example 3.9 (Primes)**.** $R = \{2,3,5,7,\ldots\}$ is not paradoxical: make the multiples of 4 true. Two multiples of 4 never differ by a prime; and if $n \equiv 1 \pmod 4$ then $n+3$, if $n \equiv 2$ then $n+2$, and if $n \equiv 3$ then $n+5$ is a multiple of 4.

Example 3.10 (Squares)**.** Whether $R = \{1,4,9,16,\ldots\}$ is paradoxical is open. A kernel would be a set no two of whose members differ by a square, hence a set of density zero by the theorem of Furstenberg (1977) and Sárközy (1978); whether such a set can also be absorbing is Problem 9.1.

[6] The paradox can also be seen directly. If S_n is true then S_{n+2} and S_{n+4} are untrue, and each can be absorbed only by its odd successor, so S_{n+3} and S_{n+5} are true; but S_{n+3} says that S_{n+5} is untrue.

[7] A Python script accompanying the paper checks each kernel claim exactly, the sets involved being eventually periodic, and searches for eventually periodic kernels with preperiod at most 4 and period at most 8 where none is claimed. The search is a sanity check, not a proof.

The powers of two and the primes are infinite sieves with valuations, while the even numbers, a sieve smaller than either in density and in growth, are paradoxical. The size of the sieve is irrelevant; only its structure matters.

A question from 2002

Jongeling et al. (2002, Example 6) consider the following system, indexed by the positive integers written in decimal:

$$(S_n) \quad \text{for every } k > n \text{ whose decimal numeral ends with that of } n,\ S_k \text{ is untrue.}$$

Thus S_7 speaks about $S_{17}, S_{27}, \ldots, S_{107}, \ldots$, and S_{17} about $S_{117}, S_{217}, \ldots$. The authors compute finite approximations, observe growing initial blocks of untrue sentences, and give no verdict. The system is paradoxical. Reverse the numerals: the numeral of k ends with that of n iff the reversed numeral of k is the reversed numeral of n followed by a non-empty string of digits that does not end in 0. The system is therefore the sieve system on the ten digits with R the set of non-empty digit strings not ending in 0, restricted to the vertices that are reversed numerals; this set of vertices is closed under appending words of R, so every vertex keeps its entire forward cone. Two strings that do not end in 0 concatenate to a string that does not end in 0, so R is a subsemigroup and Corollary 3.3 applies.[8] This appears to be the earliest tree-indexed Yablo sequence in print.

Trees are roomier

Lemma 3.4 does not survive the move to trees. On $\{0,1\}^*$ the sieve $0^+ = \{0, 00, 000, \ldots\}$ is paradoxical, since each ray $u, u0, u00, \ldots$ carries a copy of Yablo's sequence. Add one word to the sieve:

$$R = 0^+ \cup \{1\}: \qquad (S_w) \quad S_{w0}, S_{w00}, S_{w000}, \ldots \text{ and } S_{w1} \text{ are all untrue.}$$

Every sentence still sees a whole Yablo sequence, and yet the system has a valuation.

Proposition 3.11. *$\mathscr{Y}(\{0,1\}, 0^+ \cup \{1\})$ has a classical valuation.*

Proof. Call a word a *base* if it is empty or ends in 1. Every word is uniquely $u0^j$ with u a base and $j \geq 0$, and the words $u, u0, u00, \ldots$ form the *ray* of u. Any two vertices on a ray are joined by an edge, so a kernel contains at most one vertex per ray, and a kernel is described by a function k from bases to $\mathbb{N} \cup \{\infty\}$: the ray of u meets K in $u0^{k(u)}$, or not at all if $k(u) = \infty$. The kernel conditions are the following, for every base u and every $j \geq 0$, where $v = u0^j1$ is itself a base:

- if $j = k(u)$, then $v \notin K$, that is, $k(v) \neq 0$, since the vertex $u0^j \in K$ must not point into K;
- if $j > k(u)$, then $v \in K$, that is, $k(v) = 0$, since the vertex $u0^j \notin K$ sees nothing of K on its ray and must be absorbed through v;
- if $j < k(u)$, there is no condition, since $u0^j$ is absorbed by $u0^{k(u)}$.

Define $k(\varnothing) = 0$, and for a base $v = u0^j1$ let $k(v) = 0$ if $j > k(u)$ and $k(v) = 1$ otherwise. All three conditions hold by construction. □

The digraph $\mathscr{D}(\{0,1\}, 0^+ \cup \{1\})$ contains Yablo's digraph as an induced subgraph on every ray, so containing a copy of Yablo's sequence is not sufficient for a uniform system to be paradoxical, although by Rabern et al. (2013) and Hsiung (2025) something of the kind is necessary. The example is the mirror image of one given by Walicki (2019, Fig. 2.6), a kernel-less acyclic digraph that contains no subdivision of Yablo's digraph at all. Together the two examples show that the Yablo pattern is neither a subgraph condition nor an induced-subgraph condition for the absence of a kernel; this is why Beringer and Schindler (2017) formulate their conjecture in terms of finitary minors, and why

[8] Equivalently, "ends with" is serial and transitive on numerals, and the observation of Ketland (2005) applies.

Beringer (2025) observes that kernel-lessness, unlike danger, cannot be characterised by excluded subgraphs at all. Whether the digraph is *dangerous*, that is, whether some assignment of sentences other than the uniform one is paradoxical on it, is a different question. The proposition also lies outside the analysis of Schlenker (2007), under which every sentence of a series built from a shift-invariant generalised quantifier on ω must take the same truth-value: a sieve is not a generalised quantifier over the tail, and its valuations need not be constant.

4 Hidden circles

A sieve on $\mathbb{N}$ is *periodic* if membership in it depends only on the residue of the distance modulo some p: fix p and a set $C \subseteq \mathbb{Z}_p$ of residues, and let

$$R_{p,C} = \{r \geq 1 : r \bmod p \in C\}.$$

The odd sieve is $R_{2,\{1\}}$, the even sieve is $R_{2,\{0\}}$, and Yablo's sieve is $R_{1,\{0\}}$. Write $\mathrm{Cay}(\mathbb{Z}_p, C)$ for the *circulant* digraph on $\mathbb{Z}_p$ with an edge $d \to d+c$ for each $c \in C$.

Theorem 4.1 (Periodic sieves)**.** *The kernels of $\mathscr{D}(\mathbb{N}, R_{p,C})$ are exactly the sets*

$$K_A = \{n \in \mathbb{N} : n \bmod p \in A\}$$

with A a kernel of $\mathrm{Cay}(\mathbb{Z}_p, C)$. In particular $\mathscr{Y}(\mathbb{N}, R_{p,C})$ is paradoxical iff $\mathrm{Cay}(\mathbb{Z}_p, C)$ has no kernel.

Proof. First, lifting. Let A be a kernel of the circulant. If $n \in K_A$ and $r \in R_{p,C}$ then $n+r$ lies in a class $d+c$ with $d \in A$, $c \in C$, which is not in A; so K_A is independent. If $n \notin K_A$, its class d is absorbed in the circulant by some $d+c \in A$, and $n+r \in K_A$ for any $r \geq 1$ congruent to c; so K_A is absorbing.

Conversely, let K be any kernel and let A be the set of residue classes that contain infinitely many members of K. Four observations complete the proof.

(i) $A \neq \varnothing$. For K is non-empty (else nothing is absorbed), and if K were finite, any number larger than all its members would not be absorbed.

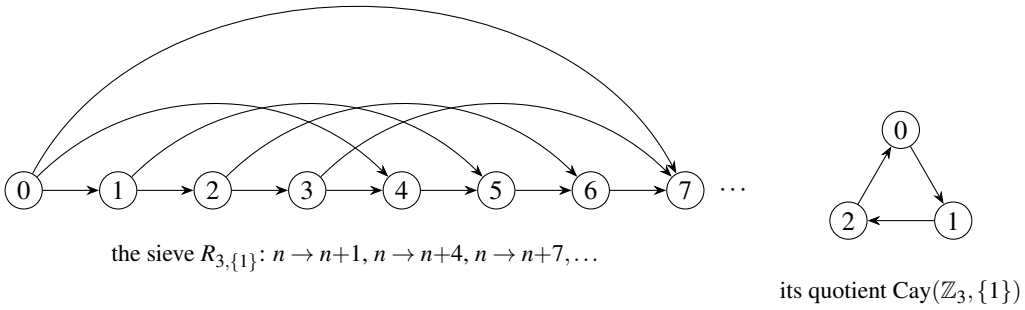
(ii) A is independent in the circulant. If $n \in K$ lies in class d, then by independence of K the classes $d+c$ ($c \in C$) contain no member of K above n, so they are finite classes: $d+C$ misses A. This holds for every class d that meets K at all, and in particular for $d \in A$.

(iii) A is absorbing in the circulant. Let $d' \notin A$. Only finitely many classes are outside A and each contains finitely many members of K; pick m in class d' larger than all of them. Then $m \notin K$, so $m+r \in K$ for some $r \in R_{p,C}$; and $m+r$, being larger than m, lies in a class belonging to A. Its class is $d'+c$ for some $c \in C$.

(iv) $K = K_A$. If $n \in K$ lies in class d, then $d+C$ misses A by (ii); if $d \notin A$ this contradicts (iii); so $K \subseteq K_A$. If n lies in a class $d \in A$, every referent $n+r$ lies in a class $d+c \notin A$, hence outside K by what was just shown; so $n \in K$.

By (ii) and (iii), A is a kernel of the circulant, and by (iv) K is its lift. □

In Cook's terms the sieve $R_{p,C}$ is an unwinding of the circulant: S_n says about every later sentence in the class $n+c$ what the vertex n mod p says about the vertex $n+c$. In the terms of Bezem et al. (2012, Theorem 3.9), whose layered lifting G^ω has vertices (d,i) with $(d,i) \to (d',j)$ iff $d \to d'$ and $i < j$, and whose kernels are exactly the ω-fold copies of the kernels of G, the sieve digraph is the subgraph of G^ω induced on the diagonal $\{(n \bmod p, n) : n \in \mathbb{N}\}$. Induced subgraphs do not inherit kernels, which is why a separate proof is needed. Hsiung (2021) proves in general that unwinding along a digraph preserves paradoxicality; what Theorem 4.1 adds is that *every* valuation of the sieve is periodic, with the period of the sieve, so that the finite digraph determines not only whether valuations exist but what they are.



the sieve $R_{3,\{1\}}$: $n \to n+1$, $n \to n+4$, $n \to n+7, \ldots$

its quotient $\mathrm{Cay}(\mathbb{Z}_3, \{1\})$

Figure 1: A directed triangle in disguise (Example 4.3). The infinite digraph on the left is acyclic and has no kernel; the finite one on the right is its quotient by residues modulo 3 and has no kernel because it is an odd cycle. By Theorem 4.1, the valuations of the one are exactly the lifts of the kernels of the other.

Consequences

Corollary 4.2. *Let $R = R_{p,C}$.*

(a) If $0 \in C$, so that R contains every positive multiple of p, the system is paradoxical. This is Lemma 3.4; in the circulant, every vertex has a loop.

(b) If $\mathrm{Cay}(\mathbb{Z}_p, C)$ has no odd directed cycle, the system is not paradoxical (Richardson, 1953). Contrapositively, every paradoxical periodic sieve has an odd directed cycle in its quotient, *where a loop counts as a cycle of length one.*

(c) If $\mathrm{Cay}(\mathbb{Z}_p, C)$ is a tournament, that is, if $C \cap (-C) = \varnothing$ and $C \cup (-C) = \mathbb{Z}_p \setminus \{0\}$, then the system is paradoxical unless some residue d is pointed to by all the others, in which case the unique valuation makes exactly the class of d true.

Proof. Parts (a) and (b) follow from the theorem and the results cited. For (c), an independent set in a tournament has at most one vertex, and $\{d\}$ is absorbing iff every other vertex points to d. □

Example 4.3 (A three-cycle in disguise)**.** $R = R_{3,\{1\}} = \{1, 4, 7, 10, \ldots\}$:

$$(S_n) \quad \text{every later sentence whose index is } n+1 \text{ modulo 3 is untrue.}$$

No element of R is absorbing and no ray is contained in R, so neither Theorem 3.2 nor Lemma 3.4 applies; but $\mathrm{Cay}(\mathbb{Z}_3, \{1\})$ is a directed triangle, a tournament with no vertex of in-degree 2, so the system is paradoxical by Corollary 4.2(c) (Figure 1). The argument can also be run directly: if S_n is true then S_{n+1} is untrue, so some $S_{n+1+3i+1} = S_{n+2+3i}$ is true, so S_{n+3+3i} is untrue, so some $S_{n+4+3i+3j}$ is true—and $n+4+3i+3j \equiv n+1$, so this is one of the sentences S_n said was untrue. Hence every S_n is untrue, and then every S_n is true. The triangle is the quotient of the reference digraph by the symmetry of the sieve, and it is the circle that the sequence was accused of hiding.

Example 4.4 (Paley sieves)**.** Let p be a prime with $p \equiv 3 \pmod 4$ and let C be the set of non-zero quadratic residues modulo p. Then -1 is a non-residue, so $C \cap (-C) = \varnothing$ and $\mathrm{Cay}(\mathbb{Z}_p, C)$ is the Paley tournament, in which every vertex has in-degree $(p-1)/2 < p-1$; Corollary 4.2(c) applies, and for $p = 3$ this is Example 4.3. The sieve

$$R = \{r \geq 1 : r \text{ is a non-zero square modulo } p\}$$

is therefore paradoxical for every such p; for $p = 7$ it is $\{1, 2, 4, 8, 9, 11, 15, 16, 18, \ldots\}$.

Sieve R	Paradoxical?	Why	Where
$\mathbb{N}^+$ (Yablo)	yes	loop in quotient; 1 absorbing	Prop. 2.3
$\{1,3,5,\dots\}$ odd	no	evens true; bipartite	Prop. 3.5
odd primes; $r \equiv 2\ (4)$	no	bipartite (2-adic valuation)	Prop. 3.5
$\{2,4,6,\dots\}$ even	yes	ray $2\mathbb{N}^+$; loop in quotient	Lem. 3.4
$\{1\} \cup 2\mathbb{N}^+$	yes	ray $2\mathbb{N}^+$	Lem. 3.4
$\mathbb{N}^+ \setminus \{2\}$	yes	3 absorbing	Thm. 3.2
$\{1,2,4,8,\dots\}$	no	multiples of 3 true	Ex. 3.8
primes	no	multiples of 4 true	Ex. 3.9
squares	?	any valuation has density 0	Prob. 9.1
$r \equiv 1 \pmod 3$	yes	directed triangle	Ex. 4.3
$r \not\equiv 0 \pmod 3$	no	multiples of 3 true	Ex. 4.5
$r \equiv 0 \pmod 3$	yes	loop in quotient	Cor. 4.2
$r \equiv 1 \pmod 4$	no	evens true; 4-cycle	Cor. 4.2
$r \equiv 1,2 \pmod 4$	yes	no kernel in $\mathrm{Cay}(\mathbb{Z}_4,\{1,2\})$	Cor. 4.2
$r \equiv \pm 1 \pmod 5$	no	$\{0,2\}$ in the 5-cycle	Cor. 4.2
$r \equiv 1,2 \pmod 5$	yes	regular tournament	Cor. 4.2
squares mod p, $p \equiv 3\ (4)$	yes	Paley tournament	Cor. 4.2
any finite R	no	compactness	Cor. 2.5
On the binary tree:			
$\{0,1\}^+$ (all descendants)	yes	0 absorbing	Cor. 3.3
$\{0,1\}$ (children only)	no	depth parity	Thm. 2.4
0^+ (left ray)	yes	each ray is Yablo	§3
$0^+ \cup \{1\}$	no	recursive valuation	Prop. 3.11
$\{0,1\}^+ \setminus \{0\}$	yes	1 absorbing	Thm. 3.2
even-length words	yes	subsemigroup	Cor. 3.3
odd-length words	no	depth parity	Prop. 3.5

Table 1: Sieves and their verdicts. The quotient of a periodic sieve is the circulant $\mathrm{Cay}(\mathbb{Z}_p, C)$ of Theorem 4.1.

Example 4.5 (Density is irrelevant)**.** The sieve $R_{3,\{1,2\}}$ of numbers not divisible by 3 has density $2/3$ and is harmless: $\mathrm{Cay}(\mathbb{Z}_3,\{1,2\})$ is the complete digraph on three vertices, every singleton is a kernel, and each residue class modulo 3 is a valuation. The sieve $R_{3,\{0\}}$ of multiples of 3 has density $1/3$ and is paradoxical.

Example 4.6 (Smaller moduli)**.** The quotient of $R_{4,\{1\}}$ is a directed four-cycle, and the even sentences form a valuation. The quotient $\mathrm{Cay}(\mathbb{Z}_4,\{1,2\})$ of $R_{4,\{1,2\}}$ has no kernel: any two of its vertices are joined one way or the other, so a kernel would be a singleton $\{d\}$, and d is pointed to only by $d-1$ and $d-2$. The sieve $\{1,2,5,6,9,10,\dots\}$ is therefore paradoxical, although it contains no ray and no absorbing element. The quotient of $R_{5,\{1,4\}}$ is an undirected five-cycle, with the kernels $\{0,2\}$, $\{1,3\}$, $\{2,4\}$, $\{3,0\}$ and $\{4,1\}$. The quotient of $R_{5,\{1,2\}}$ is a regular tournament on five vertices, not a Paley tournament since $5 \equiv 1 \pmod 4$, and it has no kernel, so the sieve $\{1,2,6,7,11,12,\dots\}$ is paradoxical.

Length-periodic sieves on trees

Call a sieve on Σ^* *length-periodic* if membership in it depends only on the length of the word modulo some p, so that $R = \{g \in \Sigma^+ : |g| \bmod p \in C\}$ for some $C \subseteq \mathbb{Z}_p$. The lifting half of Theorem 4.1 survives on trees.

Proposition 4.7. *If A is a kernel of $\mathrm{Cay}(\mathbb{Z}_p, C)$, then $K_A = \{w \in \Sigma^* : |w| \bmod p \in A\}$ is a kernel of $\mathscr{D}(\Sigma, R)$ for the length-periodic sieve R with residues C. In particular, if the circulant has a kernel, the tree system is not paradoxical.*

Proof. The proof is the lifting half of Theorem 4.1 with $|w|$ in place of n: a referent wg of $w \in K_A$ has length in a class $d + c \notin A$, and a word outside K_A, of length in class $d' \notin A$, has a referent of length in a class $d' + c \in A$, since words of every length ≥ 1 in every residue class occur in R. □

The converse half of Theorem 4.1 used the finiteness of the set of residue classes to pass to a tail of $\mathbb{N}$. A tree has no tail, and whether a length-periodic sieve whose circulant has no kernel can nevertheless have a kernel that is not a lift is open (Problem 9.2). For the triangle it cannot: the sieve of words of length $\equiv 1 \pmod 3$ is paradoxical by the direct argument of Example 4.3, with "a word of length $\equiv 1$" in place of "$3i+1$".

Sieves that are only eventually periodic, such as $\mathbb{N}^+ \setminus \{2\}$ and $\{1\} \cup 2\mathbb{N}^+$, are not covered by Theorem 4.1; they are treated in §5.

5 Deciding paradox

Theorem 4.1 settles the periodic sieves by hand. For the sieves that can be given to an algorithm at all, namely the eventually periodic subsets of $\mathbb{N}$ and the regular sets of words on a tree, the question of paradoxicality is decidable, and the decision procedure comes from automata theory. Recall that the monadic second-order theory of $(\mathbb{N}, <)$ is decidable, and that a sentence of it keeps its truth-value when its set variables are restricted to range over ultimately periodic sets (Büchi, 1962, Theorems 2 and 3); that the monadic second-order theory of the full m-ary tree with its m successor functions is decidable (Rabin, 1969, Theorem 1.12); and that a non-empty definable set of labelled trees contains a regular one, by Rabin's basis theorem (Rabin, 1972). Thomas (1990) is the standard exposition.

Theorem 5.1. *(a) For eventually periodic $R \subseteq \mathbb{N}^+$ it is decidable whether $\mathscr{Y}(\mathbb{N}, R)$ is paradoxical, and if it is not, then it has an ultimately periodic classical valuation.*

(b) For $|\Sigma| \geq 2$ and regular $R \subseteq \Sigma^+$ it is decidable whether $\mathscr{Y}(\Sigma, R)$ is paradoxical, and if it is not, then it has a classical valuation in which the set of true sentences is a regular language.

Proof. By Lemma 2.1 it suffices to express "K is a kernel of $\mathscr{D}$" as a monadic second-order formula in the set variable K, with $\mathscr{D}$ the reference digraph.

(a) Write $R = F \cup \{r \geq N_0 : r \bmod p \in C\}$ with F finite. The edge relation "$m = n + r$ for some $r \in R$" is the disjunction of "m is the f-th successor of n" over $f \in F$, which is first-order in the successor relation and hence definable from $<$, and of "$m \geq n + N_0$ and $m - n \equiv c \pmod p$ for some $c \in C$". The congruence is expressible: it holds iff there are sets $X_0, \ldots, X_{p-1}$ partitioning the interval $[n, m]$ such that $n \in X_0$, every $x \in X_i$ with $x < m$ has its successor in $X_{(i+1) \bmod p}$, and $m \in X_c$. Then "K is a kernel" is the formula $\forall n\,[\, n \in K \leftrightarrow \forall m\,(n \to m \to m \notin K)\,]$, and the sentence $\exists K\,(\ldots)$ is decidable by Büchi's theorem. If it holds, Büchi's Theorem 3 supplies an ultimately periodic set K that satisfies it, and this is an ultimately periodic kernel.

(b) Fix a finite automaton recognising R. The relation "$v = wg$ for some $g \in R$" holds iff w is a prefix of v and there is a labelling of the vertices on the path from w to v by states of the automaton that starts at the initial state, respects the transition table along each successor step, and ends in an accepting state; this is monadic second-order over the tree. The rest is as in (a), with Rabin's theorem and the basis theorem in place of Büchi's. A regular tree has finitely many distinct subtrees, and these serve as the states of a finite automaton recognising the set of vertices labelled true, so that set is a regular language. □

The procedure is not practical, since the complexity of monadic second-order logic is non-elementary; Theorem 4.1 remains the working tool. The theorem is nevertheless an island in a sea of undecidability. Salehi (2020) proves that over arbitrary structures the statement "$\mathscr{D}$ has no kernel" is not equivalent to any first-order sentence or theory, and Bezem et al. (2012) prove that for recursive digraphs, even for recursive dags, the existence of a kernel is Σ^1_1-complete, and that a recursive binary

tree may have a kernel without having a recursive one. The sieves escape because they are not arbitrary: the regularity of R places the whole problem inside monadic second-order logic, which is decidable, and the decision procedure returns a regular kernel whenever one exists.

For finite sieves there is an elementary argument that gives more.

Corollary 5.2. *If $R \subseteq \mathbb{N}^+$ is finite, then every classical valuation of $\mathscr{Y}(\mathbb{N},R)$ is purely periodic, and at least one exists.*

Proof. Let $m = \max R$ and let F_R be the map on $\{0,1\}^m$ that sends $(x_1,\dots,x_m)$ to $(f,x_1,\dots,x_{m-1})$, where $f = 1$ iff $x_r = 0$ for every $r \in R$. A 0–1 sequence (x_n) is a kernel of $\mathscr{D}(\mathbb{N},R)$ iff its windows $w_n = (x_{n+1},\dots,x_{n+m})$ satisfy $w_n = F_R(w_{n+1})$ for every n, that is, iff each window is a preimage of the one before it. In a finite set, a point with an infinite chain of preimages lies on a cycle of the map. Hence every window of a kernel lies on a cycle of F_R, so the kernel is purely periodic; the kernels correspond exactly to the periodic points of F_R; and a periodic point exists by the pigeonhole principle. □

The sieve $\{1,2\}$ has three kernels, the multiples of 3 and its two shifts, and the sieve $\{3,4,7\}$ has twenty-nine. The digraph $n \to n+r$ for r in a finite set R is the game graph of the *subtraction game* with subtraction set R (Berlekamp et al., 2001), played upward instead of downward. Played downward the game ends, its $\mathscr{P}$-positions form the unique kernel of the game graph, and they are eventually periodic (Golomb, 1966; Althöfer and Bültermann, 1995); played upward the game never ends and every position is a draw (Fraenkel, 1997). Reversing the windows conjugates F_R to the window map of the downward game, so the kernels of the upward game are exactly the time-reversals of the two-sided periodic solutions of the subtraction-game recurrence. Yablo's sieve $\mathbb{N}^+$ is the upward game in which every move is allowed.

6 In arithmetic

The sieves can be formalised exactly as Priest (1997), Ketland (2005), Leitgeb (2001), Barrio (2010), Picollo (2013), Cieśliński and Urbaniak (2013), Leach-Krouse (2014) and Halbach and Zhang (2017) formalise Yablo's sequence: a truth predicate T is added to the language of arithmetic, and the diagonal lemma supplies a formula that says of its own later instances that they are untrue. The kernel lemma then turns Ketland's analysis into a statement about every sieve at once.

Fix a finite alphabet Σ and code words over Σ by numbers, in such a way that every number is the code of a word, that concatenation $w \frown g$ is a primitive recursive function, and that the sieve R is given by an arithmetical formula $\rho(g)$; for the sieves of this paper ρ can be taken bounded. The diagonal lemma with parameters (Karimi, 2019a, Theorem 1) supplies a formula $Y_R(x)$ of the language with T such that

$$\mathrm{PA} \vdash \forall w \Big(Y_R(w) \leftrightarrow \forall g \big(\rho(g) \to \neg \mathrm{T}^{\ulcorner} Y_R(\dot{w} \frown \dot{g})^{\urcorner} \big) \Big). \tag{6}$$

The sentences $Y_R(\bar{w})$ for $w \in \Sigma^*$ are the sieve system $\mathscr{Y}(\Sigma,R)$ written in arithmetic.[9] Let

$$\mathscr{T}_R^{\mathrm{loc}} = \mathrm{PA} + (6) + \{\mathrm{T}^{\ulcorner} Y_R(\bar{w})^{\urcorner} \leftrightarrow Y_R(\bar{w}) : w \in \Sigma^*\},$$
$$\mathscr{T}_R^{\mathrm{unif}} = \mathrm{PA} + (6) + \forall w \big(\mathrm{T}^{\ulcorner} Y_R(\dot{w})^{\urcorner} \leftrightarrow Y_R(w) \big),$$

where PA is formulated in the language with T and has induction for all formulas of that language, as in Ketland (2005). The local theory disquotes each actual sentence of the sieve; the uniform theory disquotes all of them at once, non-standard ones included.

[9] Halbach and Zhang (2017) obtain the fixed point for Yablo's sequence without the diagonal lemma, from a disquotation schema for a satisfaction predicate applied to its own name together with axioms stating that the order is serial and transitive, and Karimi (2019a) weakens those axioms to the hypothesis of Theorem 3.1. The same construction yields (6) for any definable reference relation.

Theorem 6.1. *For every arithmetically definable sieve R:*

(a) $\mathscr{T}_R^{\mathrm{loc}}$ *is consistent.*

(b) $\mathscr{T}_R^{\mathrm{loc}}$ *has a model whose arithmetical part is standard iff* $\mathscr{Y}(\Sigma,R)$ *is not paradoxical.*

(c) Suppose that $\mathscr{Y}(\Sigma,R)$ *is paradoxical and that PA, with induction in the language with* T*, refutes the sentence* $\forall w\,(Y_R(w) \leftrightarrow \forall g\,(\rho(g) \to \neg Y_R(w \frown g)))$*, as it does for every paradox proved in this paper. Then* $\mathscr{T}_R^{\mathrm{loc}}$ *is* ω*-inconsistent and* $\mathscr{T}_R^{\mathrm{unif}}$ *is inconsistent.*

Proof. (a) By compactness it suffices to find, for each L, a model of PA and (6) in which disquotation holds for the finitely many words of length at most L. Take the standard model and interpret T on the codes of the sentences $Y_R(\bar{v})$ by downward recursion on the length of v: for $|v| > L$ let $\mathrm{T}^{\ulcorner} Y_R(\bar{v})^{\urcorner}$ be false, and for $|v| \le L$ let it be true iff $\mathrm{T}^{\ulcorner} Y_R(\bar{v} \frown \bar{g})^{\urcorner}$ is false for every $g \in R$, all of which are longer than v and so already decided. Interpret T arbitrarily elsewhere. Every theorem of PA holds in this expansion of the standard model, in particular (6), and (6) together with the recursion gives $Y_R(\bar{v}) \leftrightarrow \mathrm{T}^{\ulcorner} Y_R(\bar{v})^{\urcorner}$ for every v of length at most L.

(b) In a model with standard arithmetical part, the set $K = \{w : Y_R(w) \text{ holds}\}$ satisfies, by (6) and local disquotation, $w \in K$ iff $wg \notin K$ for all $g \in R$; so K is a kernel of $\mathscr{D}(\Sigma,R)$. Conversely, a kernel K yields an interpretation of T on the standard model, true at $\ulcorner Y_R(\bar{w})^{\urcorner}$ iff $w \in K$ and true at the true arithmetical sentences, which satisfies $\mathscr{T}_R^{\mathrm{loc}}$.

(c) Under the hypothesis, in any model of $\mathscr{T}_R^{\mathrm{loc}}$ the set defined by Y_R is not a kernel, so disquotation fails somewhere: $\mathscr{T}_R^{\mathrm{loc}}$ proves $\exists w \neg(\mathrm{T}^{\ulcorner} Y_R(\dot{w})^{\urcorner} \leftrightarrow Y_R(w))$, while it proves the negation of each instance. That is ω-inconsistency; and $\mathscr{T}_R^{\mathrm{unif}}$ asserts the universal closure that $\mathscr{T}_R^{\mathrm{loc}}$ refutes. □

For $R = \mathbb{N}^+$ this is Ketland's theorem. The sieve turns it into a dial: for the odd sieve or the primes, $\mathscr{T}_R^{\mathrm{loc}}$ has standard models, in which the even instances of Y_R, or those at multiples of four, are true; for $\{1,4,7,\dots\}$ it has none, because a directed triangle has no kernel. By (b), the ω-inconsistency that Ketland (2005) identified as the formal residue of Yablo's paradox is exactly the absence of a kernel on the standard structure, and the absence of a kernel is what §§3–5 compute.

Remark 6.2. On the hypothesis of (c). The proof of Theorem 3.2 needs no induction at all. The proof of Lemma 3.4 needs the fact that a definable set with at most g elements is bounded, which is an instance of induction for a formula containing T; this is why induction was extended to the full language. The proof of Theorem 4.1 speaks of residue classes containing infinitely many members of K; inside a model of PA this becomes "unboundedly many", the number of classes is standard, and the argument goes through. Theorem 5.1 is a decision procedure rather than a proof, and for a sieve settled only by running it one would have to check separately that PA can follow the automaton. A sieve with no standard models but a consistent uniform theory would be one whose kernel-lessness is true but not provable in PA for the formula Y_R (Problem 9.4).

7 Self-similarity

An *iterated function system* is a finite family of contractions $f_i : X \to X$, indexed by $i \in \Sigma$, of a complete metric space X. Hutchinson (1981) showed that there is a unique non-empty compact set $\mathscr{K} \subseteq X$ with

$$\mathscr{K} = \bigcup_{i \in \Sigma} f_i(\mathscr{K}), \tag{7}$$

the *attractor* of the system; the middle-thirds Cantor set, the Sierpiński gasket and the Koch curve are the attractors of systems with two, three and four maps respectively (Barnsley, 1988; Mandelbrot, 1982). Iterating (7) gives the *cells* $\mathscr{K}_w = f_w(\mathscr{K})$ for $w = i_1 \cdots i_n \in \Sigma^*$, where $f_w = f_{i_1} \circ \cdots \circ f_{i_n}$. The cells below w are the cells of a copy of $\mathscr{K}$ shrunk into $\mathscr{K}_w$, and under the strong separation condition the map $w \mapsto \mathscr{K}_w$ is a bijection from Σ^* onto the cells, with $w \sqsubset v$ iff $\mathscr{K}_v \subseteq \mathscr{K}_w$.

Put a sentence on each cell, and let it speak about the cells reached from it by the maps in a sieve $R \subseteq \Sigma^+$:

$$(S_{\mathscr{K}_w}) \quad \text{for every } g \in R, \text{ the sentence on } f_g(\mathscr{K}_w) \text{ is untrue.} \tag{8}$$

Since $f_g(\mathscr{K}_w) = \mathscr{K}_{wg}$, this system is $\mathscr{Y}(\Sigma, R)$, and the results of §§3–5 apply to it. On the Sierpiński gasket, the system in which every cell says that every sub-cell is untrue is paradoxical (Corollary 3.3); the system in which every cell speaks only of its three immediate sub-cells is not (Corollary 2.5); the sieve of even depths is paradoxical (Corollary 3.3) and the sieve of odd depths is not (Proposition 3.5). The geometry of $\mathscr{K}$ plays no part in any of this. The gasket, a three-map Cantor set and the full ternary tree have the same address monoid and therefore the same systems, and the overlaps that complicate some attractors identify infinite addresses, that is, points, whereas the sentences sit on finite addresses.[10]

Two fixed points

Hutchinson's theorem is Banach's fixed-point theorem applied to the operator $\mathscr{F}(A) = \bigcup_i f_i(A)$ on the space of compact sets: the attractor is the unique solution of $A = \mathscr{F}(A)$, and it is self-similar because it is a fixed point of an operator built from copies of itself. The diagonal lemma with parameters produces a formula $Y(x)$ with

$$\mathrm{PA} \vdash \forall x \left(Y(x) \leftrightarrow \forall y > x \, \neg \mathrm{T}^{\ulcorner} Y(\dot{y})^{\urcorner} \right),$$

which is a fixed point of the operator "say of every later instance of $\cdot$ that it is untrue". This fixed point is what Priest (1997) identified as the circularity of Yablo's sequence: not any one sentence, but the formula that generates all of them, refers to itself. If that is circularity, then the Sierpiński gasket is circular in the same sense, and so is every self-similar structure. The sieve systems are the fixed points of the operator "say of every R-image of $\cdot$ that it is untrue" on labelled trees, the attractors are the fixed points of the Hutchinson operator on compact sets, and the paradox proofs of §3 are renormalisation arguments, in which an assumption about one cell is pushed down to a smaller copy and contradicted there. The analogy ends exactly where it should. Hutchinson's operator is a contraction and has one fixed point; the diagonal lemma's operator has many, one for each sieve among others, and nothing forces the fixed point it returns to admit a classical valuation. The paradox lives in that gap.

Limit points

The points of the attractor are the limits of cells along infinite branches, that is, the elements of Σ^ω, and the sentences never reach them. Give each point $x \in \Sigma^\omega$ a sentence of its own, and let the cells speak about points as well as cells:

$$(S_w) \quad \text{every cell below } w \text{ and every point in } \mathscr{K}_w \text{ carries an untrue sentence,}$$
$$(S_x) \quad \text{(nothing: a point has nothing below it).}$$

The point sentences are sinks and therefore true; every cell contains a point, so every cell sentence is untrue, correctly; the system has a classical valuation, and only one. This is Proposition 2.3 again: the fractal supplies a last word on every branch, and the paradox evaporates just as it does on $\omega + 1$. Whether one is entitled to a sentence about a point of a fractal is a question about truth rather than about fractals, and it is not pursued here. The observation is only that the usual remedy for an ungrounded sequence, namely something at the limit for it to be about, is in this setting the passage from the tree to its boundary.

[10] If the sentences are put on cells as sets rather than on addresses, and two maps have the same image, then distinct addresses carry one sentence and the monoid is no longer free. Theorem 3.2 survives, since its proof uses only that $w \mapsto wg$ is a map with $gR \subseteq R$. What could go wrong is a loop $wg = w$, and a non-degenerate system of contractions has none, because a contraction with $f^k = f^{k+t}$ has f^k constant.

8 Circularity

Priest (1997) observed that the sequence is given by a single predicate $Y(x)$, "no $k > x$ satisfies Y", which the diagonal lemma supplies as a fixed point of an operator built from Y itself, and that finite reasoners have nothing else to go on, since they do not apply the ω-rule. Sorensen (1998) replied that a circular description can pick out a non-circular object and that an infinite mind could survey the list directly; Beall (2001) replied that until the reference of "Yablo's sequence" is fixed otherwise than by Priest's description, the sequence under discussion is the circular one. Bueno and Colyvan (2003a,b) tried to derive a contradiction from a stipulated list with the local truth-biconditionals alone, and Ketland (2004, 2005) showed that this cannot be done, because the list with local disquotation has non-standard models, so that the contradiction needs the uniform fixed point together with uniform disquotation. Leitgeb (2002) found both the reference-based and the fixed-point-based notions of circularity wanting, and noted that under Aczel's anti-foundation axiom the Liar's set $\Omega = \{\Omega\}$ and an infinite descending chain are one and the same set. Cook (2006, 2014) answered Priest that weak fixed points are everywhere in arithmetic and that the infinitary version of the sequence has none; Butler (2018) built Yabloesque sequences whose satisfaction conditions are not recursive, so that Priest's circularity could not be recognised in them; Picollo (2012) proposed a notion of reference on which the list is not circular although the principle generating it is; Ojea (2012) found an anti-foundation axiom, Boffa's, under which the collapse onto the Liar does not occur; Çevik (2020) proposed that the paradox is "circular in the limit"; and Yablo (2006) observed that a Tarskian hierarchy stops the Liar but not the ω-Liar unless it grounds out in a bottom language.[11] The results above allow three things to be said that bear on this debate.

Circles are the finite case

In the reference-based sense of circularity, a sentence is circular if it refers to itself through a chain of references; this is the sense of Leitgeb (2002), of Picollo's m-reference, and of Ojea's "some node reaches itself". In this sense none of the systems in this paper is circular, and some of them are paradoxical. For *finite* uniform systems, however, circularity is exactly the right diagnosis. By Richardson's theorem the only obstruction to a kernel in a finite digraph is an odd directed cycle, so a finite uniform system is paradoxical only if some sentence refers to itself through a chain of odd length. A chain of even length does not suffice: the no-no pair is a two-cycle, and two-cycles have kernels. Walicki (2023a) locates here the difference between the reference-graph tradition of Rabern et al. (2013), in which every directed cycle is dangerous because a reference graph does not record where the negations sit, and the kernel tradition, in which only odd cycles do harm; Walicki (2017, §6) puts it as the slogan that a finitary paradox needs negative self-reference. The diagnosis fails only where Theorem 2.4(a) fails, namely where some sentence has infinitely many referents.

The circle is in the quotient

For periodic sieves Priest's charge can be made exact without any mention of predicates or diagonalisation. The sieve $R_{p,C}$ is generated by the finite digraph $\mathrm{Cay}(\mathbb{Z}_p, C)$, of which the infinite system is the unwinding; by Theorem 4.1 the infinite system is paradoxical iff the finite digraph has no kernel; and by Richardson's theorem a finite digraph without a kernel has an odd cycle. Every paradoxical periodic sieve therefore has an odd cycle in its quotient, and its sentences are the non-circular unwinding of a circular finite system. For Yablo's own sieve the quotient is a single vertex with a loop, that is, the

[11] There is more: Hardy (1995) and Tennant (1995); Goldstein (1994) and Yablo (2002) on the set-theoretic pattern; Bringsjord and van Heuveln (2003) on the "mental eye" defence; Forster (2004) on ill-founded type hierarchies; Urbaniak (2009) on aboutness; Landini (2008), who denies that there is a paradox at all; Barrio (2012), who surveys the symposium on Cook's book; and Visser (2002), who in work from 1989 independently answered Kripke's question about descending hierarchies of languages.

Liar; for the sieve of distances $\equiv 1 \pmod 3$ it is a directed triangle, the three-sentence Liar cycle whose unwinding Cook (2004) already knew to be paradoxical. The circularity that Priest located in the generating predicate is, for these systems, the circularity of the generating digraph, and it is visible to any reasoner, finite or not.

The quotient does not always do this work. Length-periodic tree sieves have a quotient, and it certifies valuations (Proposition 4.7); the quotient of the full-tree sieve Σ^+ by the parity of length is a pair of vertices with loops, whose odd cycles do witness the paradox. But a quotient is a homomorphic image, and kernel-lessness does not pass down homomorphisms; Schlechta (2018) refuted a conjecture of Rabern et al. (2013) on exactly this point, and the harmless sieve $0^+ \cup \{1\}$ of Proposition 3.11 collapses, under length modulo any p, onto a digraph with a loop at every vertex. What the paradoxical tree sieves of Theorem 3.2 have in common is not a cycle in a quotient but a copy of the whole below every vertex, together with the absorption property. Self-similarity, not circularity, is the structural fact, and self-similarity is what a fixed-point description records, of a sieve as of the Sierpiński gasket. If a fixed point counts as a circle then, as Cook (2006) says, almost everything is circular; and if the sieve is not recursive then, as Butler (2018) says, the circle cannot be seen. Nothing in §§3–5 required R to be recursive.

The structural collapse

Under Aczel's anti-foundation axiom the identity criterion for sets is bisimulation of the pointed membership graph from a chosen vertex, and the existence of a kernel is not a bisimulation invariant: the one-point loop has no kernel, while the two-cycle is bisimilar to it and has two. Every serial digraph is bisimilar to the one-point loop. The collapse that Leitgeb (2002) described, that Yablo (2006) seems to have found congenial and that Cook (2014) rejects, therefore identifies the odd sieve with Yablo's sieve and with the Liar, although one of the three has a classical valuation and the other two do not. A representation of reference structure that forgets kernels forgets what a paradox is. The observation of Ojea (2012) that Boffa's axiom preserves paradoxicality where Aczel's does not is the observation that Boffa's identity criterion is finer than bisimulation. For periodic sieves the right invariant is the kernel-lessness of the quotient (Theorem 4.1); for regular sieves it is at least decidable (Theorem 5.1).

Verdict

In the sense in which the Liar is circular, namely that some sentence refers, through others, to itself, Yablo's sequence is not circular, and neither is any of its relatives in this paper. In the sense in which the Sierpiński gasket is circular, namely that the structure is the fixed point of an operator that builds it from copies of itself, Yablo's sequence is circular, and so are the harmless relatives. The second sense does not distinguish paradox from its absence; the first distinguishes them only among finite systems. What distinguishes them in general is the existence of a kernel, and for the self-similar systems of this paper that question comes down, wherever it can be computed, to an odd cycle in a finite quotient or an absorbing element in the sieve. The paradox needs no circle; but where the sequence has a finite shadow, the shadow has one.

9 Problems

Open problem 9.1 (Squares)**.** Is the sieve of squares (Example 3.10) paradoxical? A classical valuation would be a square-difference-free set that is also upward absorbing, in the sense that every non-member has a member a square above it. The first condition forces density zero (Furstenberg, 1977; Sárközy, 1978), and the known sparse square-difference-free sets (Ruzsa, 1984) are not obviously

absorbing. The same question arises for the cubes, and for any sieve of polynomial values to which the generalisations of the Furstenberg–Sárközy theorem apply.

Open problem 9.2 (Trees)**.** On trees there is a sufficient condition for paradox (Theorem 3.2), a sufficient condition for its absence (Proposition 4.7), an example showing that a full Yablo ray below every vertex is not enough (Proposition 3.11), and a decision procedure for regular sieves (Theorem 5.1). Is a length-periodic tree sieve paradoxical whenever its circulant has no kernel? More generally, is there a finite-digraph criterion for regular sieves, obtained from a quotient of $\mathscr{D}(\Sigma, R)$ by the automaton recognising R whose kernels lift?

Open problem 9.3 (Random sieves)**.** Let R contain each positive integer independently with probability $q \in (0,1)$. Is $\mathscr{Y}(\mathbb{N}, R)$ almost surely paradoxical, almost surely not, or neither? The greedy construction, which takes 0 and then repeatedly the least number whose differences from the members chosen so far lie outside R, produces an infinite independent set $K = \{k_1 < k_2 < \cdots\}$ almost surely, growing roughly like $(1-q)^{-i}$. For a fixed $n \notin K$, absorption fails only if $k_i - n \notin R$ for every i; if the bits $R(k_i - n)$ were independent coins of bias q this would have probability zero, and there are only countably many n. The bits are not independent of the construction, however, and a proof would have to address that. If the expectation is right, then a random sieve is harmless, with almost all of its sentences untrue in every valuation, and the paradoxical sieves are the structured ones: rays, semigroups, and odd cycles in the quotient.

Open problem 9.4 (Formalisation)**.** By Theorem 6.1, a sieve with no classical valuation whose kernel-lessness is not provable in PA for the formula Y_R would have a consistent uniform theory $\mathscr{T}_R^{\mathrm{unif}}$: a Yablo-style sequence whose paradoxicality is true but unprovable. Does such a sieve exist among the arithmetically definable ones? The squares would be a candidate if Problem 9.1 were settled affirmatively by a non-constructive argument.

Open problem 9.5 (Signs)**.** A *signed* sieve is a set $R \subseteq \Sigma^+ \times \{+,-\}$, with S_w saying that S_{wg} is true for $(g,+) \in R$ and untrue for $(g,-) \in R$; the interleaved systems of Proposition 2.2 are the simplest examples. Through the graph normal form of Bezem et al. (2012) and Walicki (2017), in which a positive reference $x \to y$ becomes $x \to m \to y$ with a fresh vertex m saying only that y is untrue, a signed system is a uniform system on a digraph with the positive edges subdivided once, and paradox is governed by the parity of paths, as Walicki's "3-Yablo" and the signed reference graphs of Beringer and Schindler (2017) illustrate. Do Theorems 3.2 and 4.1 have signed analogues? For the second, replacing the kernels of $\mathrm{Cay}(\mathbb{Z}_p, C)$ by the solutions of the finite signed system is essentially Cook's unwinding theorem; for the first, subdividing the positive edges destroys the domination $N^+(wg) \subseteq N^+(w)$ on which Theorem 3.1 rests.

Open problem 9.6 (Non-classical valuations)**.** In the least fixed point of Kripke (1975) every sentence of every sieve system is undefined, since no sentence has an empty set of referents, so groundedness does not distinguish the sieves at all. The other fixed points do: Yablo's sieve has only the empty one, while the odd sieve has exactly three, the empty one and the two parity valuations, because a single determinate sentence forces the whole parity pattern. What is the lattice of fixed points of a periodic sieve, and is it determined by the circulant, as the classical valuations are? Hsiung (2017) relates finite Boolean paradoxes to the periods of their revision sequences; how are the period of a sieve and the period of its revision related?

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